

CHARACTERS OF REAL SPECIAL 2-GROUPS

DILPREET KAUR AND AMIT KULSHRESTHA

ABSTRACT. It is well-known that special 2-groups can be described in terms of quadratic maps over fields of characteristic 2. In this article we develop methods to compute conjugacy classes, complex representations and characters of a real special 2-group *using quadratic maps alone*.

1. INTRODUCTION

Special 2-groups are the 2-groups for which the commutator subgroup, the Frattini subgroup and the centre, all three coincide and are isomorphic to an elementary abelian group. A particular case is that of *extraspecial 2-groups* where, in addition, the centre is required to be of order 2. Non-abelian groups of order 8 and their central products are examples of extraspecial 2-groups [7, §3.10.2]. Special 2-groups can be described in terms of quadratic maps between vector spaces over the field of order 2 [8, §1.3].

A group G is called a *real group* if for each $x \in G$, the conjugacy classes of x and x^{-1} are same. A *strongly real group* is the one in which every element can be expressed as a product of at most two elements of order 2. Every strongly real group is real.

Recently special 2-groups have been studied to establish that there are infinitely many strongly real groups which admit complex symplectic representations, and vice-versa, there are infinitely many groups which are not strongly real and do not admit symplectic representations [4]. This generates an interest in the computation of conjugacy classes, representations and character table of special 2-groups. In this article we explore the description of special 2-groups as quadratic maps to make these computations for real special 2-groups. Our methods to compute representations, characters and conjugacy classes can be implemented directly on the quadratic maps associated to special 2-groups. These methods are based on the understanding of representations of extraspecial 2-groups. The key point of our proofs lies in the demonstration that the representations of extraspecial 2-groups can indeed be patched together to construct all representations of real special 2-groups. This is done by converting quadratic maps to quadratic forms by composing them with suitable linear maps. A crucial step is a refinement in a result of Zahinda [8,

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Prop. 3.3].

The article is organized as follows: In §2 we define quadratic maps, special 2-groups and recall the connection between them. We also describe how one can construct quadratic forms from these quadratic maps and utilize the understanding of extraspecial 2-groups to construct all complex representations of real special 2-groups, up to equivalence. Then in §3 and §4 we describe representations and characters of real special 2-groups in terms of representations and characters of extraspecial 2-groups. A computation of conjugacy classes of special 2-groups is done in §4.2. We conclude the article with §5 by an explicit computation of character table of a particular real special 2-group following the methods developed in preceding sections.

The main results of this article are Theorems 3.12, 4.4, 4.7 and 4.8. Theorem 3.12 describes all non-linear irreducible representations of real special 2-groups, Theorem 4.4 describes all irreducible characters and Theorems 4.7 and 4.8 describe conjugacy classes of these groups. Detailed statements of these theorems carry quite a bit of notation, and we avoid that in the introductory section.

2. QUADRATIC MAPS AND SPECIAL 2-GROUPS

Throughout this article \mathbb{F} denotes a field of characteristic 2 and \mathbb{F}_2 denotes the field with two elements. Let V and W be vector spaces over \mathbb{F} . A map $q : V \rightarrow W$ is called a *quadratic map* if $q(\alpha v) = \alpha^2 q(v)$ for all $v \in V$, for all $\alpha \in \mathbb{F}$ and the map $b_q : V \times V \rightarrow W$ defined by $b_q(v, w) = q(v + w) - q(v) - q(w)$ is bilinear. The bilinear map b_q is called the *polar map* associated to the quadratic map q . We denote by $\langle b_q(V \times V) \rangle$ the \mathbb{F} -subspace of W generated by the image of b_q . The set of quadratic maps between V and W is denoted by $\text{Quad}(V, W)$. The subspace $\text{rad}(b_q) := \{v \in V : b_q(v, w) = 0 \ \forall w \in V\}$ of V is called the *radical* of (V, q) . A quadratic map q is said to be *regular* if $\text{rad}(b_q) = 0$. A quadratic map is called a *quadratic form* if $W = \mathbb{F}$. For the properties of quadratic forms over a field of characteristic 2, we refer to [2] and [5].

For a group G , let $Z(G)$ denote its centre and $[G, G]$ denote its derived subgroup. Let $\Phi(G)$ be the Frattini subgroup of G . A 2-group G is called a *special 2-group* if $\Phi(G) = [G, G] = Z(G) \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^n$ for some $n \in \mathbb{N}$. Note that these conditions imply that the quotient $\frac{G}{Z(G)}$ and the centre $Z(G)$ both are elementary abelian 2-groups [1, Ch. 5, Th. 1.3]. In what follows, we describe special 2-groups through quadratic maps.

2.1. Quadratic maps associated to special 2-groups. We briefly recollect the connection between special 2-groups and quadratic maps. For details we refer to [8, §1.3]. Let G be a special 2-group. Let $W := Z(G)$ and $V := \frac{G}{Z(G)}$. Since W and V are elementary abelian 2-groups, we may regard them as vector spaces over \mathbb{F}_2 . Consider

the map $q : \frac{G}{Z(G)} \rightarrow Z(G)$ defined by $q(xZ(G)) = x^2$ for $xZ(G) \in \frac{G}{Z(G)}$. The map q is a well defined quadratic map and its *polar map* $b_q : V \times V \rightarrow W$ is given by $b_q(xZ(G), yZ(G)) = xyx^{-1}y^{-1}$, where $x, y \in G$. As a quadratic map q is *regular* and $\langle b_q(V \times V) \rangle = W$ [4, Lemma 2.3]. The map q is called the *quadratic map associated to the special 2-group* G . Conversely, the following theorem asserts that a regular quadratic map between two finite dimensional \mathbb{F}_2 -vector spaces uniquely defines a special 2-group provided $\langle b_q(V \times V) \rangle = W$.

Theorem 2.1 ([8], Th. 1.4). *Let V and W be two finite dimensional vector spaces over \mathbb{F}_2 and $q : V \rightarrow W$ be a regular quadratic map. Let b_q be the polar map of q . If $\langle b_q(V \times V) \rangle = W$ then there exists a special 2-group G such that the quadratic map associated to G is q . Such a group is unique up to isomorphism.*

Let V and W be finite dimensional vector spaces over \mathbb{F}_2 . In what follows, we regard V as an additive group acting trivially on W . In this set up we consider the group $Z^n(V, W)$ of n -cocycles of V with coefficients in W and its subgroup $B^n(V, W)$ of n -coboundaries. The n^{th} cohomology group $H^n(V, W)$ for the trivial action of V on W is the quotient of $Z^n(V, W)$ by $B^n(V, W)$. Our interest in this article is restricted to the case $n = 2$. Every element in $H^2(V, W)$ can be represented by a normal 2-cocycle. By definition, a *normal 2-cocycle* is a map $c : V \times V \rightarrow W$ satisfying following conditions:

- (1) $c(v_2, v_3) - c(v_1 + v_2, v_3) + c(v_1, v_2 + v_3) - c(v_1, v_2) = 0$
- (2) $c(v, 0) = c(0, v) = 0$

where $v, v_1, v_2, v_3 \in V$. The image in $H^2(V, W)$ of a normal 2-cocycle $c \in Z^2(V, W)$ is denoted by $[c]$.

Proposition 2.2 ([8], Prop. 1.2). *Let V and W be two vector spaces over \mathbb{F}_2 and $c : V \times V \rightarrow W$ be a normal 2-cocycle. Then the map $q_c : V \rightarrow W$ defined by $q_c(x) = c(x, x)$ is a quadratic map. If V is finite dimensional over \mathbb{F}_2 then the correspondence $q_c \longleftrightarrow [c] \in H^2(V, W)$ is a bijection between $\text{Quad}(V, W)$ and $H^2(V, W)$.*

Remark 2.3. The bijection $q_c \longleftrightarrow [c]$ between $\text{Quad}(V, W)$ and $H^2(V, W)$ as in Prop. 2.2 is denoted by ϕ . It allows us to associate a special 2-groups to quadratic maps. The explicit description is as follows. Let $q : V \rightarrow W$ be a finite dimensional quadratic map satisfying $\langle b_q(V \times V) \rangle = W$. Let c be a normal 2-cocycle on V with coefficients in W such that $\phi(q) = [c]$. On the Cartesian product $V \times W$, consider the binary operation defined by

$$(v, w) \cdot (v', w') = (v + v', c(v, v') + w + w')$$

for all $v, v' \in V$ and $w, w' \in W$. An easy calculation shows that the above binary operation defines a group structure on $V \times W$. We denote this group by G_q . The identity element of G_q is $(0, 0)$ and the inverse of (v, w) in G_q is $(v, c(v, v) + w)$. The group G_q turns out to be a special 2-group and the quadratic map associated to G_q is q . It is called the *group associated to the quadratic map* $q : V \rightarrow W$. We shall drop the subscript q from the notation G_q whenever the description of q is evident.

In view of Remark 2.3, we fix the following notation for elements of special 2-groups defined by a quadratic map $q : V \rightarrow W$.

Type of element	Notation
Arbitrary element	$(v, w) \in V \times W = G$
Central element	$(0, w) \in W = Z(G)$
Element in $\frac{G}{Z(G)}$	$(v, 0) \in V = \frac{G}{Z(G)}$

2.2. Extraspecial 2-groups. Special 2-groups whose centre is of order 2 are called *extraspecial 2-groups*. They play an important role in the understanding of representations of real special 2-groups. The structure of extraspecial 2-groups is well understood. To describe that we need to define the notion of central product. Let H and K be two finite groups with isomorphic centers and $\zeta : Z(H) \rightarrow Z(K)$ be a group isomorphism. A *central product* $G = H \circ_{\zeta} K$ of H and K with respect to the isomorphism ζ is defined to be the quotient $\frac{H \times K}{N}$ where $N := \langle \{(h, k) \in Z(H) \times Z(K) : \zeta(h)k^{-1} = 1\} \rangle$. If the isomorphism ζ is evident in a context then we drop the subscript ζ and denote the central product by $H \circ K$.

Remark 2.4. Since the order of center of an extraspecial 2-group is 2, the quadratic map q associated to an extraspecial 2-group is a quadratic form. The quadratic form associated to the extraspecial 2-group D_4 , the dihedral group of order 8, is given by $q_1(x, y) = xy$ and the one associated to Q_2 , the quaternion group of order 8, is given by $q_2(x, y) = x^2 + xy + y^2$. Following the notation of [2] we write $q_1 = [0, 0]$ and $q_2 = [1, 1]$. From the classification of regular quadratic forms it follows that for each $n \in \mathbb{N}$ there are exactly two extraspecial 2-groups of order 2^{2n+1} , namely $D_4 \circ D_4 \circ \cdots \circ D_4$ (n copies of D_4) and $Q_2 \circ D_4 \circ \cdots \circ D_4$ ($n - 1$ copies of D_4) (see [7, §3.10.2]). Here $H \circ K$ denotes the central product of two groups H and K with isomorphic centers.

2.3. From extraspecial to special. Let G be a special 2-group and $q : V \rightarrow W$ be the quadratic map associated to G . For non-zero $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ we denote $s_*(q) := s \circ q : V \rightarrow \mathbb{F}_2$. Indeed $s_*(q)$ is a quadratic form and its polar form is $b_{s_*(q)} := s \circ b_q : V \times V \rightarrow \mathbb{F}_2$. The form $s_*(q)$ is called the *transfer of q by s* . If the radical $\text{rad}(b_{s_*(q)})$ vanishes under $s_*(q) : V \rightarrow \mathbb{F}_2$ then $s_*(q)$ induces a quadratic form $q_s : V_s := \frac{V}{\text{rad}(b_{s_*(q)})} \rightarrow \mathbb{F}_2$ as follows: $q_s(\epsilon_s(v)) = s_*(q)(v)$, $v \in V$; where $\epsilon_s : V \rightarrow V_s$ denotes the canonical surjection.

Lemma 2.5. *Let G be a special 2-group and q be the quadratic map associated to G . Suppose that the radical $\text{rad}(b_{s_*(q)})$ vanishes under $s_*(q) : V \rightarrow \mathbb{F}_2$. Then the quadratic form $q_s : V_s \rightarrow \mathbb{F}_2$ is regular and the polar map $b_{q_s} : V_s \times V_s \rightarrow \mathbb{F}_2$ is surjective.*

Proof. We first show that the quadratic form $q_s : V_s \rightarrow \mathbb{F}_2$ is regular. For $v, w \in V$, we compute

$$\begin{aligned} b_{q_s}(\epsilon_s(v), \epsilon_s(w)) &= q_s(\epsilon_s(v)) + q_s(\epsilon_s(w)) - q_s(\epsilon_s(v) + \epsilon_s(w)) \\ &= s(q(v)) + s(q(w)) - s(q(v + w)) \\ &= s(q(v) + q(w) - q(v + w)) \\ &= s(b_q(v, w)) \\ &= b_{s_*(q)}(v, w) \end{aligned}$$

Let $\epsilon_s(v) \in \text{rad}(b_{q_s})$. Then from the above computation, we conclude that $b_{s_*(q)}(v, w_i) = 0$ for some set $\{w_i\}_{i=1}^r$ of coset representatives of V in $\text{rad}(b_{s_*(q)})$. Let $w \in V$ be any arbitrary element. Then we write $w = w_i + w'$ for a suitable $w' \in \text{rad}(b_{s_*(q)})$ and $1 \leq i \leq r$. Then

$$b_{s_*(q)}(v, w) = b_{s_*(q)}(v, w_i) + b_{s_*(q)}(v, w') = 0$$

Therefore $v \in \text{rad}(b_{s_*(q)})$ and $\epsilon_s(v) = 0$. Thus $\text{rad}(b_{q_s})$ is the trivial subspace of V_s and the quadratic form q_s is regular.

As G is special 2-group, by [4, Lemma 2.3], $\langle b_q(V \times V) \rangle = W$. Since $s : W \rightarrow \mathbb{F}_2$ is non-zero, $b_{q_s}(V_s \times V_s) = s(b_q(V \times V)) = \mathbb{F}_2$. \square

With all notations as above, we consider the quadratic form $q_s : V_s \rightarrow \mathbb{F}_2$. By Lemma 2.5 and Theorem 2.1 there exists a special 2-group G_s such that the quadratic map associated to G_s is q_s . We denote by c_s a normal 2-cocycle such that $\phi(q_s) = [c_s]$ (see Prop. 2.2 and Remark 2.3).

Remark 2.6. If G is a real special 2-group then $s_*(q)(\text{rad}(b_{s_*(q_G)})) = 0$. We refer to [4, §3] for details. Recall that a group G is called *real* if for each $x \in G$, the conjugacy classes of x and x^{-1} are same.

We now record results on real special 2-groups that will be useful later in the article. The following result characterizes real special 2-groups in terms of the associated quadratic map.

Theorem 2.7 ([8], Th. 2.1). *Let G be a special 2-group and $q : V \rightarrow W$ be the quadratic map associated to G . The following assertions are equivalent:*

- i. *The group G is real.*
- ii. *For all $v \in V$, there exists $v' \in V$ such that $q(v') = q(v + v')$.*

The following result for real special 2-groups is implicit in the proof of [8, Prop. 3.3]

Lemma 2.8. *Let G be a real special 2-group and $q : V \rightarrow W$ be the quadratic map associated to G . For $0 \neq s \in \text{Hom } \mathbb{F}_2(W, \mathbb{F}_2)$ let $q_s : V_s \rightarrow \mathbb{F}_2$ be the quadratic form as in Lemma 2.5. Let G_s denote the special 2-group associated to q_s . Then*

- i. The group G_s is extraspecial 2-group.
- ii. $V_s \simeq \frac{G_s}{Z(G_s)}$.

Proof. i. Since the quadratic form q_s associated to the special 2-group G_s takes values in \mathbb{F}_2 , it follows that $|Z(G_s)| = 2$. Therefore G_s is extraspecial.

ii. We recall from Remark 2.3 that the underlying set of G_s is the Cartesian product $V_s \times \mathbb{F}_2$ and its group operation is given by $(v, w)(v', w') = (v + v', c_s(v, v') + w + w')$, where $v, v' \in V_s$; $w, w' \in \mathbb{F}_2$. Here c_s is a normal 2-cocycle whose cohomology class corresponds to the quadratic form q_s (see Prop. 2.2). Regarding V_s as an abelian group under addition, we consider the group homomorphism $\xi : G_s \rightarrow V_s$ given by $\xi(v, w) = v$. Clearly ξ is a surjection. Therefore $\frac{G_s}{\ker \xi} \cong V_s$. Now to prove the result, we need to show that $\ker \xi = Z(G_s)$. If $(v, w) \in \ker \xi$ then $v = 0$. Thus for all $(v', w') \in G_s$, we have

$$(0, w)(v', w') = (0 + v', c_s(0, v') + w + w') = (v' + 0, c_s(v', 0) + w + w') = (v', w')(0, w)$$

This confirms that $Z(G_s)$ contains $\ker \xi$. Now for the reverse inclusion, let $(v, w) \in Z(G_s)$. For all $(v', w') \in G_s$, we have

$$\begin{aligned} (v, w)(v', w') &= (v', w')(v, w) \\ \Rightarrow (v + v', c_s(v, v') + w + w') &= (v' + v, c_s(v', v) + w' + w) \\ \Rightarrow c_s(v, v') &= c_s(v', v). \end{aligned}$$

By [8, Prop. 1.4], we have $b_{q_s}(v, v') = c_s(v, v') - c_s(v', v)$. From the above calculation, we have $b_{q_s}(v, v') = 0$ for all $v' \in V_s$. Thus $v \in \text{rad}(b_{q_s})$. Since q_s is regular quadratic form, $v = 0$ and therefore $(v, w) \in \ker \xi$.

□

3. REPRESENTATIONS OF REAL SPECIAL 2-GROUPS

In this section we describe irreducible representations of real special 2-groups. We begin with representations of degree one.

3.1. Linear representations of special 2-groups. Throughout this article, by a *linear representation* we mean a representation of degree one. With this nomenclature, representations of degree at least two will be called *non-linear representations*. Finding linear representations of special 2-groups is elementary and is based on the following well-known results.

Theorem 3.1 ([3], Th. 17.11). *Let G be a finite group. The linear representations of G are precisely the lifts to G of the irreducible representations of $\frac{G}{[G, G]}$. In particular, the number of distinct linear representations of G equals the index of $[G, G]$ in G .*

Let $A = [a_{ij}]_{n \times n}$ and B be two matrices over a field. Then the tensor product of matrices A and B is defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}$$

The following theorem provides a description of representations of direct products of groups in terms of irreducible representations of direct factors.

Theorem 3.2 ([1], Ch. 3, Th. 7.1). *Let H and K be two finite groups and $G = H \times K$ be the direct product of H and K . Let $\rho : H \rightarrow \text{GL}(n, \mathbb{C})$ and $\sigma : K \rightarrow \text{GL}(m, \mathbb{C})$ be irreducible representations of H and K , respectively. Then $\rho \otimes \sigma : G \rightarrow \text{GL}(nm, \mathbb{C})$ defined by $(\rho \otimes \sigma)(h, k) = \rho(h) \otimes \sigma(k)$ for $(h, k) \in H \times K = G$ is an irreducible representation of G . Moreover, every irreducible representation of $G = H \times K$ is equivalent to a representation of the form $\rho \otimes \sigma$ for a suitable choice of ρ and σ .*

Remark 3.3. For a special 2-group G the quotient $\frac{G}{Z(G)}$ is isomorphic to a direct product of copies of $\frac{\mathbb{Z}}{2\mathbb{Z}}$. Clearly $\frac{\mathbb{Z}}{2\mathbb{Z}}$ has only one non-trivial irreducible representation. Thus from Theorem 3.2 one can write all irreducible representations of $\frac{G}{Z(G)}$. Now using Theorem 3.1 and the equality $Z(G) = [G, G]$ for special 2-groups, one can write all linear representations of special 2-groups.

Interesting part of the discussion is to describe non-linear irreducible representations of special 2-groups. In this article we limit ourselves to real special 2-groups. For a real special 2-group G we shall describe non-linear irreducible representations of G in terms of non-linear irreducible representations of extraspecial 2-groups G_s . The section 3.2 concerns the representations of extraspecial 2-groups and the section 3.3 concerns the non-linear representations of real special 2-groups.

3.2. Non-linear representations of extraspecial 2-groups. Let H and K be two finite groups with isomorphic centers. Let $\zeta : Z(H) \rightarrow Z(K)$ be a group isomorphism. Let $N := \langle \{(h, k) \in Z(H) \times Z(K) : \zeta(h)k^{-1} = 1\} \rangle$ and $G = H \circ K \cong \frac{H \times K}{N}$ be the central product of groups H and K . Let $\varphi : H \times K \rightarrow \text{GL}(n, \mathbb{C})$ be a representation of $H \times K$ such that $N \subseteq \ker(\varphi)$. Then φ can be treated as a representation $\widehat{\varphi}$ of G simply by defining $\widehat{\varphi}(\overline{(h, k)}) = \varphi(h, k)$, where $\overline{(h, k)} = (h, k)N$ for all $(h, k) \in H \times K$. The representation φ is irreducible if and only if $\widehat{\varphi}$ is irreducible [1, Ch. 3, Th. 7.2]. Since extraspecial 2-groups are central products of copies of non-abelian groups of order 8, we repeatedly use the constructions like $\widehat{\varphi}$ to describe all non-linear irreducible representations of extraspecial 2-groups.

Remark 3.4.

- (1) For the dihedral group $D_4 = \langle a, b : a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ the homomorphism $\rho : D_4 \rightarrow \text{GL}(2, \mathbb{C})$ defined by

$$\rho(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the only non-linear irreducible representation.

- (2) For the quaternion group $Q_2 = \langle c, d : c^4 = 1, d^2 = c^2, dcd^{-1} = c^{-1} \rangle$ the homomorphism $\sigma : Q_2 \rightarrow \text{GL}(2, \mathbb{C})$, where

$$\sigma(c) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma(d) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

is the only non-linear irreducible representation.

Based on this remark, we have the following :

Proposition 3.5. *Every extraspecial 2-group has a unique non-linear representation.*

Proof. Let $\gamma_i : 1 \leq i \leq 4$ be linear representations of the group D_4 . From Theorem 3.2 the non-linear irreducible representation of the group $D_4 \times D_4$ are $\rho \otimes \gamma_i : 1 \leq i \leq 4$ and $\rho \otimes \rho$, where ρ is as in Remark 3.4(1). The normal subgroup $N := \{(1, 1), (a^2, a^2)\}$ of the group $D_4 \times D_4$ is contained in the kernel of $\rho \otimes \rho$ but N is not contained in the kernel of $\rho \otimes \gamma_i$ for all $1 \leq i \leq 4$. As $D_4 \circ D_4 = \frac{D_4 \times D_4}{N}$, from Theorem [1, Ch. 3, Th. 7.2] it is evident that $\widehat{\rho \otimes \rho}$ is the only non-linear representation of $D_4 \circ D_4$, which is induced from the representation $\rho \otimes \rho$ of $D_4 \times D_4$. This generalizes to the fact the representation $\rho \otimes \widehat{\rho \otimes \cdots \otimes \rho}$ (l copies of ρ) is the only non-linear representation of $D_4 \circ D_4 \circ \cdots \circ D_4$ (l copies of D_4). Similarly the group $Q_8 \circ D_4 \circ \cdots \circ D_4$ ($l-1$ copies of D_4) has unique non-linear irreducible representation, namely $\sigma \otimes \widehat{\rho \otimes \cdots \otimes \rho}$ ($l-1$ copies of ρ). \square

Remark 3.6. The degree of unique non-linear representation of extraspecial 2-group of order 2^{2l+1} is 2^l .

The following theorem is well-known standard result.

Theorem 3.7 ([3], Th. 11.12). *Let G be a finite group and $\chi_1, \chi_2, \dots, \chi_k$ be the complete list of distinct irreducible characters of G . Then $\sum_{i=1}^k \chi_i(1)^2 = |G|$.*

Lemma 3.8. *Let G be an extraspecial 2-group of order 2^{2l+1} and χ be the character of its unique non-linear irreducible representation. Then*

$$\chi(g) = \begin{cases} 2^l & \text{if } g \text{ is the identity element of } G \\ -2^l & \text{if } g \text{ is a non-trivial element of } Z(G) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Remark 2.4, the group G is isomorphic to either $D_4 \circ D_4 \circ \cdots \circ D_4$ (l copies of D_4) or $Q_2 \circ D_4 \circ D_4 \circ \cdots \circ D_4$ ($l-1$ copies of D_4). Let ρ and σ be unique non-linear representations of D_4 and Q_2 , respectively, as in Remark 3.4. Let $\varphi : G \rightarrow \text{GL}(2^l, \mathbb{C})$ be the non-linear irreducible representation of G . Then by Prop. 3.5, $\varphi = \rho \otimes \widehat{\rho \otimes \cdots \otimes \rho}$ (l copies of ρ) if $G \cong D_4 \circ D_4 \circ \cdots \circ D_4$ (l copies of D_4) or $\varphi = \sigma \otimes \widehat{\rho \otimes \cdots \otimes \rho}$ ($l-1$ copies of ρ) if $G \cong Q_2 \circ D_4 \circ D_4 \circ \cdots \circ D_4$ ($l-1$ copies of D_4). If $G \cong D_4 \circ D_4 \circ \cdots \circ D_4$ (l copies of D_4), using Theorem 3.2, we know that $\varphi(\bar{1}) = \rho(1) \otimes \rho(1) \otimes \cdots \otimes \rho(1)$ (l times). Therefore $\chi_\varphi(\bar{1}) = \text{tr}(\rho(1) \otimes \rho(1) \otimes \cdots \otimes \rho(1))$, where χ_φ is character associated to representation φ . Using Remark 3.4 and the fact that for two matrices A and B , $\text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B)$, we get $\chi_\varphi(\bar{1}) = 2^l$. If $\bar{1} \neq g \in Z(G)$ then $g = \overline{(a^2, 1, \dots, 1)}$, where a^2 is the non-trivial element of $Z(D_4)$. Then we have

$$\begin{aligned} \chi_\varphi(g) &= \text{tr}(\rho(a^2) \otimes \rho(1) \otimes \cdots \otimes \rho(1)) \\ &= \text{tr}(\rho(a^2)) \text{tr}(\rho(1)) \cdots \text{tr}(\rho(1)) \\ &= -2^l \end{aligned}$$

If $g = \overline{(g_1, g_2, \dots, g_l)} \in G \setminus Z(G)$, then for some $1 \leq i \leq l$, we have $g_i \in D_4 \setminus Z(D_4)$ and $\rho(g_i) = 0$. Thus $\chi_\varphi(g) = 0$. This proves the result if $G \cong D_4 \circ D_4 \circ \cdots \circ D_4$ (l copies of D_4). One can prove the result for $G \cong Q_2 \circ D_4 \circ D_4 \circ \cdots \circ D_4$ ($(l-1)$ copies of D_4) on similar lines. \square

3.3. Non-linear representations of real special 2-groups. In this section we describe all non-linear irreducible representations of real special 2-groups. We begin with recalling the following.

Proposition 3.9 ([8], Prop. 3.3). *Let G be a real special 2-group and $q : V := \frac{G}{Z(G)} \rightarrow Z(G) =: W$ be the quadratic map associated to G . Then*

- (1) *For every non-zero $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$, there exists a non-linear irreducible representation φ of G such that $\varphi(G) = G_s$.*
- (2) *Conversely, for all non-linear irreducible representations φ of G , there exists a non-zero $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ such that $\varphi(G) = G_s$.*

We refine the first part of Prop. 3.9 by observing that for every non-zero $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$, there are exactly $|\text{rad}(b_{s_*(q)})|$ many of inequivalent non-linear irreducible representations of G such that $\varphi(G) = G_s$. Here $|\text{rad}(b_{s_*(q)})|$ denotes the size of the radical $\text{rad}(b_{s_*(q)})$.

Before stating next proposition, we record some definitions, which will be used later.

- (1) Let $c : V \times V \rightarrow W$ be a normal 2-cocycle and $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$. Then $s_*(c) : V \times V \rightarrow \mathbb{F}_2$ defined by $s_*(c)(v, v') = s(c(v, v'))$ for all $v, v' \in V$ is a normal 2-cocycle. It is called the *transfer of c by s* .

- (2) Recall from §2.3 that $V_s = \frac{V}{\text{rad}(b_{s_*(q)})}$. Let $\epsilon_s : V \rightarrow V_s$ be the canonical surjection and $c_s : V_s \times V_s \rightarrow W$ be a normal 2-cocycle. Then $\text{Inf}(c_s) : V \times V \rightarrow \mathbb{F}_2$ defined by $\text{Inf}(c_s)(v, v') = c_s(\epsilon_s(v), \epsilon_s(v'))$ for $v, v' \in V$ is a normal 2-cocycle. It is called the *inflation* of c_s .

Proposition 3.10. *Let G be a real special 2-group and $q : V := \frac{G}{Z(G)} \rightarrow Z(G) =: W$ be the quadratic map associated to G . Then for every non-zero $s \in \text{Hom } \mathbb{F}_2(W, \mathbb{F}_2)$ there exists at least $|\text{rad}(b_{s_*(q)})|$ many surjective homomorphisms from G to the extraspecial 2-group G_s .*

Proof. Let $s \in \text{Hom } \mathbb{F}_2(W, \mathbb{F}_2)$ be a non-zero map. Since $\text{rad}(b_{s_*(q)})$ is a subspace of V , we have $|\text{rad}(b_{s_*(q)})| = 2^k$ for some $k \in \mathbb{N}$. Since the order of $\text{Hom } \mathbb{F}_2(\text{rad}(b_{s_*(q)}), \mathbb{F}_2)$ is same as that of $\text{rad}(b_{s_*(q)})$, we have 2^k linear maps from $\text{rad}(b_{s_*(q)})$ to \mathbb{F}_2 . We enumerate these linear maps as $\{t_i\}$; $1 \leq i \leq 2^k$. For rest of the proof we fix a vector space complement V' of $\text{rad}(b_{s_*(q)})$ in V . Thus we write $V = \text{rad}(b_{s_*(q)}) \oplus V'$. Define $h_i : V \rightarrow \mathbb{F}_2$ by $h_i(v) = t_i(x)$, where $v = (x, y) \in V$ with $x \in \text{rad}(b_{s_*(q)})$ and $y \in V'$.

Recall that there exists a normal 2-cocycle c such that $\phi(q) = [c]$, where ϕ is the isomorphism between $H^2(V, W)$ and $\text{Quad}(V, W)$ as in Prop. 2.2. Let $s \in \text{Hom } \mathbb{F}_2(W, \mathbb{F}_2)$ and $s_*(c) : V \times V \rightarrow \mathbb{F}_2$ be the transfer of c by s . Let c_s be a normal 2-cocycle such that $\phi(q_s) = [c_s]$, where ϕ is the isomorphism between $H^2(V_s, \mathbb{F}_2)$ and $\text{Quad}(V_s, \mathbb{F}_2)$ as defined in Prop. 2.2. Now we have

$$\text{Inf}(c_s)(v, v) = q_s(\epsilon_s(v)) = s(q(v)) = s_*(c)(v, v)$$

Thus both $[\text{Inf}(c_s)]$ and $[s_*(c)]$ are preimages of the same quadratic map under the isomorphism $\phi : H^2(V, \mathbb{F}_2) \rightarrow \text{Quad}(V, \mathbb{F}_2)$ given in Prop. 2.2. Therefore $\text{Inf}(c_s)$ and $s_*(c)$ are cohomologous. Thus there exists a map $\lambda : V \rightarrow \mathbb{F}_2$ such that $\lambda(0) = 0$ and

$$(1) \quad \text{Inf}(c_s)(v, v') = s_*(c)(v, v') - \lambda(v + v') + \lambda(v) + \lambda(v').$$

We are now ready to define surjective homomorphisms from G to G_s for each i ; $1 \leq i \leq 2^k$. Consider $f_{s,i} : G \rightarrow G_s$ defined by

$$f_{s,i}(v, w) = (\epsilon_s(v), s(w) - \lambda(v) - h_i(v))$$

for $v \in V, w \in W$. Here for group elements of G we follow the notation as in the table of §2.1. That each $f_{s,i} : G \rightarrow G_s$ is a homomorphism follows from the following direct computation.

$$\begin{aligned} f_{s,i}((v, w)(v', w')) &= f_{s,i}(v + v', c(v, v') + w + w') \\ &= (\epsilon_s(v + v'), s(c(v, v') + w + w') - \lambda(v + v') - h_i(v + v')) \\ &= (\epsilon_s(v) + \epsilon_s(v'), c_s(\epsilon_s(v), \epsilon_s(v')) + s(w) + s(w') - \lambda(v) - \lambda(v') - h_i(v) - h_i(v')) \\ &= (\epsilon_s(v), s(w) - \lambda(v) - h_i(v))(\epsilon_s(v'), s(w') - \lambda(v') - h_i(v')) \\ &= f_{s,i}((v, w))f_{s,i}((v', w')) \end{aligned}$$

for $(v, w), (v', w') \in G$.

To check the surjectivity, let $(v_s, w_s) \in G_s$ where $v_s \in V_s$ and $w_s \in \mathbb{F}_2$. Since maps ϵ_s and s are surjective, there exist $v \in V$ and $w \in W$ such that $\epsilon_s(v) = v_s$ and $s(w) = w_s$. Further, since $\lambda(v), h_i(v) \in \mathbb{F}_2$, there exist $w_1, w_2 \in W$ such that $\lambda(v) = s(w_1)$ and $h_i(v) = s(w_2)$. The following calculation confirms that $f_{s,i}(v, w + w_1 + w_2) = (v_s, w_s)$.

$$\begin{aligned} f_{s,i}(v, w + w_1 + w_2) &= (\epsilon_s(v), s(w + w_1 + w_2) - \lambda(v) - h_i(v)) \\ &= (\epsilon_s(v), s(w) + s(w_1) + s(w_2) - \lambda(v) - h_i(v)) \\ &= (v_s, w_s) \end{aligned}$$

Since for $i \neq j$ there exists $v \in V$ such that $h_i(v) \neq h_j(v)$, we have $f_{s,i} \neq f_{s,j}$. Thus $f_{s,i}; 1 \leq i \leq 2^k$ are distinct homomorphisms. These are $|\text{rad}(b_{s_*(q)})| = 2^k$ in number. \square

Proposition 3.11. *Let G be a real special 2-group and $q : V := \frac{G}{Z(G)} \rightarrow Z(G) =: W$ be the quadratic map associated to G . Then for every non-zero $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ there exist at least $|\text{rad}(b_{s_*(q)})|$ many inequivalent non-linear irreducible representations φ of G such that $\varphi(G) = G_s$.*

Proof. Let $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ be a non-zero linear map and k be such that $|\text{rad}(b_{s_*(q)})| = 2^k$. Let φ_s denote the unique non-linear irreducible faithful representation of the extraspecial 2-group G_s . Then $\varphi_{s,i} := \varphi_s \circ f_{s,i}$ for $1 \leq i \leq 2^k$, where $f_{s,i}$ is as in the proof of Prop. 3.10, are irreducible representations of G of the degree same as that of φ_s . Since $f_{s,i} : G \rightarrow G_s$ is surjective and φ_s is faithful, we have $\varphi_{s,i}(G) = \varphi_s(f_{s,i}(G)) = \varphi_s(G_s) \cong G_s$. We claim that $\varphi_{s,i}$ and $\varphi_{s,j}$ are equivalent if and only if $i = j$. Let $\chi_{s,i}$ and $\chi_{s,j}$ be the characters of representations $\varphi_{s,i}$ and $\varphi_{s,j}$, respectively. Then $\chi_{s,i} = \chi_s f_{s,i}$ and $\chi_{s,j} = \chi_s f_{s,j}$. Recall from [6, p. 16, Cor. 2] that $\varphi_{s,i}$ and $\varphi_{s,j}$ are equivalent if and only if $\chi_{s,i}(v, w) = \chi_{s,j}(v, w)$ for every $(v, w) \in G$. We use it to complete the proof.

If $|\text{rad}(b_{s_*(q)})| = 1$ then there is nothing to prove. If $|\text{rad}(b_{s_*(q)})| > 1$, then for $i \neq j$ there exists $(v, 0) \in G$, where $v \in \text{rad}(b_{s_*(q)})$, such that $h_i(v) \neq h_j(v)$. Thus

$$f_{s,i}(v, 0) = (0, -\lambda(v) - h_i(v)), \quad f_{s,j}(v, 0) = (0, -\lambda(v) - h_j(v))$$

Here $h_i : V \rightarrow \mathbb{F}_2$ is the homomorphism as defined in the proof of Prop. 3.10.

It follows that one of $f_{s,i}(v, 0)$ and $f_{s,j}(v, 0)$ is the identity element of the group G_s , while the other is the non-trivial element of $Z(G_s)$. Without loss of generality we assume that

$$f_{s,i}(v, 0) = (0, 0) \in Z(G_s), \quad f_{s,j}(v, 0) = (0, 1) \in Z(G_s).$$

Let the order of G_s be 2^{2l+1} . Then by Lemma 3.8

$$\begin{aligned}\chi_{s,i}(v, 0) &= \chi_s f_{s,i}(v, 0) = \chi_s(0, 0) = 2^l \\ \chi_{s,j}(v, 0) &= \chi_s f_{s,j}(v, 0) = \chi_s(0, 1) = -2^l\end{aligned}$$

This proves that $\varphi_{s,i}$ and $\varphi_{s,j}$ are inequivalent if $i \neq j$. \square

The following theorem describes all non-linear irreducible representations of real special 2-groups.

Theorem 3.12. *Let G be a real special 2-group and $q : V := \frac{G}{Z(G)} \rightarrow Z(G) =: W$ be the quadratic map associated to G . Then $\{\varphi_{s,i} : s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2), 1 \leq i \leq 2^k\}$ is the complete list of non-linear irreducible representations of G ; where $\varphi_{s,i}$ are as in Prop. 3.11 and 2^k is the size of the radical $\text{rad}(b_{s^*(q)})$.*

Proof. Let $|G| = 2^n$ and $|Z(G)| = 2^m$. The number of non-zero linear maps from $Z(G)$ to \mathbb{F}_2 is $|Z(G)| - 1 = 2^m - 1$. We denote these linear maps by $s_1, s_2, \dots, s_{2^m-1}$. First we prove that $\varphi_{s_p,i} \sim \varphi_{s_q,j}$ if either $p \neq q$ or $i \neq j$. If $p = q$ then it follows from the proof of Prop. 3.11 that $\varphi_{s_p,i} \sim \varphi_{s_q,j}$ if $i \neq j$. Suppose $p \neq q$. Recall that $\varphi_{s_p,i}(G) \cong G_{s_p}$ and $\varphi_{s_q,j}(G) \cong G_{s_q}$. If $G_{s_p} \not\cong G_{s_q}$ then $\varphi_{s_p,i} \sim \varphi_{s_q,j}$. Suppose that $G_{s_p} \cong G_{s_q}$ and $|G_{s_p}| = |G_{s_q}| = 2^{2l+1}$. Since $s_p \neq s_q$, there exist $w \in W$ such that $s_p(w) \neq s_q(w)$. Then $f_{s_p,i}(0, w) = (0, s_p(w))$ and $f_{s_q,j}(0, w) = (0, s_q(w))$. Now by the same argument as in the proof of Prop. 3.11, without loss of generality, we assume that $\chi_{s_p,i}(0, w) = 2^l$ and $\chi_{s_q,j}(0, w) = -2^l$. Here $\chi_{s_p,i}$ and $\chi_{s_q,j}$ are characters associated with representations $\varphi_{s_p,i}$ and $\varphi_{s_q,j}$ respectively.

To prove the theorem we use Theorem 3.7 and show that squares of degrees of representations $\varphi_{s,i}$ and linear representations add up to 2^n .

Let $|G_s| = 2^{2l+1}$ for some non-zero linear map $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$. Since G_s are extraspecial 2-groups, we have $|Z(G_s)| = 2$. Using Lemma 2.8(2), we have

$$\frac{|\frac{G}{Z(G)}|}{|\frac{G_s}{Z(G_s)}|} = |\text{rad } b_{s^*(q)}|$$

Therefore $|\text{rad } b_{s^*(q)}| = \frac{2^{n-m}}{2^{2l}}$. By Theorem 3.11, there are at least $|\text{rad } b_{s^*(q)}|$ number of non-linear irreducible inequivalent representations φ of G such that $\varphi(G) \cong G_s$. The degree of these representations is same as the degree of the faithful representation φ_s of G_s . We know that the degree of the non-linear faithful representation of extraspecial 2-groups of order 2^{2l+1} is 2^l (see Remark 3.6). Now we compute the sum of squares of

degrees of irreducible representations of G .

$$\begin{aligned}
\left| \frac{G}{Z(G)} \right|.1^2 + \sum_{j=1}^{2^m-1} |\text{rad } b_{s_{j^*}(q)}|. (2^{l_j})^2 &= \left| \frac{G}{Z(G)} \right|.1^2 + \sum_{j=1}^{2^m-1} \frac{|\frac{G}{Z(G)}|}{|\frac{G_{s_j}}{Z(G_{s_j})}|}. (2^{l_j})^2 \\
&= 2^{n-m} + \sum_{j=1}^{2^m-1} \frac{2^{n-m}}{2^{2l_j}}. (2^{l_j})^2 \\
&= 2^{n-m} + \sum_{j=1}^{2^m-1} 2^{n-m} \\
&= 2^{n-m} + (2^m - 1)2^{n-m} \\
&= 2^n \\
&= |G|.
\end{aligned}$$

Therefore G can not afford non-linear representations except $\varphi_{s,i}$ and $\{\varphi_{s,i} : s \in \text{Hom } \mathbb{F}_2(W, \mathbb{F}_2), 1 \leq i \leq 2^k\}$ is the complete list of non-linear irreducible representations of G . \square

4. CHARACTER TABLE OF REAL SPECIAL 2-GROUPS

The aim of this section is to provide a method to write the character table of real special 2-groups using quadratic maps alone.

4.1. Characters of real special 2-groups. In what follows, we keep the notations of Prop. 3.11.

Proposition 4.1. *Let G be a real special 2-group. Let $\chi_{s,i}$ be the character of the representation $\varphi_{s,i}$, as described in Prop. 3.12 Let the order of G_s be 2^{2l+1} . Then*

$$\chi_{s,i}(g) = \begin{cases} 2^l & \text{if } f_{s,i}(g) = 1 \\ -2^l & \text{if } f_{s,i}(g) \text{ is the non-trivial element of } Z(G_s) \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let φ_s be the non-linear irreducible representation of the extraspecial 2-group G_s and χ_s be the character of φ_s . Since $\varphi_{s,i} = \varphi_s \circ f_{s,i}$, we have $\chi_{s,i} = \chi_s \circ f_{s,i}$. Now the proof follows immediately from Lemma 3.8. \square \square

Remark 4.2. Let $\text{diag}(1, 1, \dots, 1)$ denote the identity matrix of order 2^l and $\text{diag}(-1, -1, \dots, -1)$ denote the diagonal matrix of order 2^l with diagonal entries equal to -1 . Then

$$\varphi_{s,i}(g) = \begin{cases} \text{diag}(1, 1, \dots, 1) & \text{if } \chi_{s,i}(g) = 2^l \\ \text{diag}(-1, -1, \dots, -1) & \text{if } \chi_{s,i}(g) = -2^l \end{cases}$$

Definition 4.3. For the character $\chi_{s,i}$, we define

$$\text{sign}(\chi_{s,i}(g)) = \begin{cases} -1 & \text{if } \chi_{s,i}(g) = -2^l \\ 1 & \text{if } \chi_{s,i}(g) = 2^l \end{cases}$$

Theorem 4.4. Let G be a real special 2-group and $q : V \rightarrow W$ be the quadratic map associated to G . Let $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$. Then

- (1) $\chi_{s,i}(v, w) \neq 0$ if and only if $v \in \text{rad}(b_{s_*(q)})$.
- (2) For $(0, w) \in Z(G)$ we have

$$\chi_{s,i}(0, w) = \begin{cases} 2^l & \text{if } s(w) = 0 \\ -2^l & \text{if } s(w) = 1 \end{cases}$$

where l is defined by $|G_s| = 2^{2l+1}$.

- (3) Let $\{v_1, v_2, \dots, v_k\}$ be a basis of $\text{rad}(b_{s_*(q)})$. Then

$$\chi_{s,i}(v_j, 0) = \begin{cases} -2^l & \text{if } A_{j,i} = 1 \\ 2^l & \text{if } A_{j,i} = 0 \end{cases}$$

where $A_{j,i}$ denotes the coefficient of 2^j in the binary expansion $i - 1 = \sum_{j=0}^{k-1} A_{j,i} 2^j$.

- (4) Let $g \in G$ be an element with $g = \prod_{j=1}^r (v_{i_j}, 0)(0, w)$ for $1 \leq i_1 < i_2 < \dots < i_r \leq k$ then

$$\chi_{s,i}(g) = \prod_{j=1}^r \text{sign}(\chi_{s,i}(v_{i_j}, 0)) \cdot \text{sign}(\chi_{s,i}(0, w)) \cdot 2^l$$

Proof. (1) We have $\chi_{s,i} = \chi_s \circ f_{s,i}$, where χ_s is the irreducible non-linear character of the extraspecial 2-group G_s and $f_{s,i}$ is the homomorphism as defined in Prop. 3.10. Let $(v, w) \in G$, then

$$\begin{aligned} \chi_{s,i}(v, w) &= \chi_s(f_{s,i}(v, w)) \\ &= \chi_s(\epsilon_s(v), s(w) - \lambda(v) - h_i(v)) \end{aligned}$$

By Lemma 3.8, $\chi_s(x, y) \neq 0$ if and only if $x = 0$. Thus $\chi_{s,i}(v, w) = 0$ if and only if $\epsilon_s(v) = 0$. This happens precisely when $v \in \text{rad}(b_{s_*(q)})$. Hence the result follows.

- (2) Let $(0, w) \in Z(G)$ then

$$\begin{aligned} \chi_{s,i}(0, w) &= \chi_s(f_{s,i}(0, w)) \\ &= \chi_s(\epsilon_s(0), s(w) - \lambda(0) - h_i(0)) \\ &= \chi_s(0, s(w)) \end{aligned}$$

Let l be defined by $|G_s| = 2^{2l+1}$. From Lemma 3.8 and the above calculation, we have

$$\chi_{s,i}(0, w) = \begin{cases} 2^l & \text{if } s(w) = 0 \\ -2^l & \text{if } s(w) = 1 \end{cases}$$

- (3) Let $\{v_1, v_2, \dots, v_k\}$ be a basis of $\text{rad}(b_{s_*(q)})$. Let $A_{j,i}$ be defined by the binary expansion

$$i - 1 = A_{0,i}2^0 + A_{1,i}2^1 + \dots + A_{k-1,i}2^{k-1},$$

where $1 \leq i \leq 2^k$. Consider a map $\lambda : V \rightarrow \mathbb{F}_2$ as in equation 1 in section 3.3. We define a map $\theta : \text{rad}(b_{s_*(q)}) \rightarrow \mathbb{F}_2$ by

$$\theta \left(\sum_{j=1}^r v_{i_j} \right) = \sum_{j=1}^r \lambda(v_{i_j}) \text{ for } 1 \leq i_1 < i_2 < \dots < i_r \leq k$$

Notice that the map θ is nothing but the linear extension to $\text{rad}(b_{s_*(q)})$ of the restriction of λ to the basis $\{v_1, v_2, \dots, v_k\}$. Thus $\theta \in \text{Hom}_{\mathbb{F}_2}(\text{rad}(b_{s_*(q)}), \mathbb{F}_2)$. We recall from the proof of Prop. 3.10 the description of maps $h_i : V \rightarrow \mathbb{F}_2$; $1 \leq i \leq 2^k$. By definition, $h_i(v_j) = t_i(v_j)$ as $v_j \in \text{rad}(b_{s_*(q)})$. Let $t'_i \in \text{Hom}_{\mathbb{F}_2}(\text{rad}(b_{s_*(q)}), \mathbb{F}_2)$ be defined by $t'_i(v_j) = A_{j,i}$. Since both $\{t_i : 1 \leq i \leq 2^k\}$ and $\{\theta - t'_i : 1 \leq i \leq 2^k\}$ denote the same set, namely the set of all the linear maps from $\text{rad}(b_{s_*(q)})$ to \mathbb{F}_2 , by a suitable permutation of indices we may assume that $t_i = \theta - t'_i$. Thus we have

$$\begin{aligned} \chi_{s,i}(v_j, 0) &= \chi_s(f_{s,i}(v_j, 0)) \\ &= \chi_s(\epsilon_s(v_j), s(0) - \lambda(v_j) - h_i(v_j)) \\ &= \chi_s(0, -\lambda(v_j) - t_i(v_j)) \\ &= \chi_s(0, -\lambda(v_j) - \theta(v_j) + t'_i(v_j)) \\ &= \chi_s(0, -\lambda(v_j) - \lambda(v_j) + t'_i(v_j)) \\ &= \chi_s(0, t'_i(v_j)) \\ &= \chi_s(0, A_{j,i}) \end{aligned}$$

Therefore

$$\chi_{s,i}(v_j, 0) = \begin{cases} \chi_s(0, 1) & \text{if } A_{j,i} = 1 \\ \chi_s(0, 0) & \text{if } A_{j,i} = 0 \end{cases} = \begin{cases} -2^l & \text{if } A_{j,i} = 1 \\ 2^l & \text{if } A_{j,i} = 0 \end{cases}$$

- (4) Let $\{v_1, v_2, \dots, v_k\}$ be a basis of $\text{rad}(b_{s_*(q)})$. Take $g = \prod_{j=1}^r (v_{i_j}, 0)(0, w) \in G$; where $1 \leq i_1 < i_2 < \dots < i_r \leq k$ then

$$\begin{aligned} \varphi_{s,i}(g) &= \varphi_{s,i} \left(\prod_{j=1}^r (v_{i_j}, 0)(0, w) \right) \\ &= \prod_{j=1}^r \varphi_{s,i}(v_{i_j}, 0) \varphi_{s,i}(0, w) \end{aligned}$$

Now from Prop. 4.4(2) and Prop. 4.4(3) the value of $\chi_{s,i}(v_j, 0)$ and $\chi_{s,i}(0, w)$ is either 2^l or -2^l . From Remark 4.2, we have that $\varphi_{s,i}(v_j, 0)$ and $\varphi_{s,i}(0, w)$ are either $\text{diag}(1, 1, \dots, 1)$ or $\text{diag}(-1, -1, \dots, -1)$. Now from above calculation and definition 4.3, we have

$$\chi_{s,i} \left(\prod_{j=1}^r (v_{i_j}, 0)(0, w) \right) = \prod_{j=1}^r \text{sign}(\chi_{s,i}(v_{i_j}, 0)) \cdot \text{sign}(\chi_{s,i}(0, w)) \cdot 2^l.$$

Hence the result follows. \square

We summarize Prop. 4.4 in the following table. Notations of table are same as Prop. 4.4. Let $|G| = 2^n$, $|Z(G)| = 2^m$ and $|G_s| = 2^{2l+1}$. We recall from the proof of Theorem 3.12 that in this case $|\text{rad}(b_{s_*(q)})| = 2^{n-m-2l}$. We fix an ordered basis $\{v_1, v_2, \dots, v_k\}$ of $|\text{rad}(b_{s_*(q)})|$, where $k = n - m - 2l$.

Table A

Type of element	$\chi_{s,i}(v, w)$	Number of elements
$\{(v, w) : v \notin \text{rad}(b_{s_*(q)})\}$	0	$2^n - 2^{n-2l}$
$\{(0, w) : s(w) = 0\}$ $\{(0, w) : s(w) = 1\}$	2^l -2^l	2^m
$\{(v_j, 0) : A_{j,i} = 1\}$ $\{(v_j, 0) : A_{j,i} = 0\}$ $\{(v_j, w) : 0 \neq (0, w) \in Z(G)\}$	-2^l 2^l $\text{sign}(\chi_{s,i}(v_j, 0)) \cdot \text{sign}(\chi_{s,i}(0, w)) \cdot 2^l$	$(n - m - 2l) \cdot 2^m$
$\prod_{j=1}^r (v_{i_j}, 0)(0, w)$ where $1 \leq i_1 < i_2 < \dots < i_r \leq k$ and $r \geq 2$	$\prod_{j=1}^r \text{sign}(\chi_{s,i}(v_{i_j}, 0)) \cdot \text{sign}(\chi_{s,i}(0, w)) \cdot 2^l$	$(2^{n-m-2l} - (n - m - 2l) - 1) \cdot 2^m$

4.2. Conjugacy classes of real special 2-groups. In this section we form conjugacy classes of a real special 2-group G . We frequently use the following well-known result to distinguish two conjugacy classes.

Proposition 4.5 ([3], Prop. 15.5). *Let G be a finite group and $\chi_1, \chi_2, \dots, \chi_k$ be the complete set of inequivalent irreducible characters of G . Then two elements $g, h \in G$ are conjugates if and only if $\chi_i(g) = \chi_i(h)$ for all $1 \leq i \leq k$.*

Remark 4.6. Let G be a real special 2-group and $q : V \rightarrow W$ be the quadratic map associated to G . The elements of $W = Z(G)$ form conjugacy classes containing only one element. Let $v_1 \neq v_2 \in V$ and $v_1 \neq 0, v_2 \neq 0$ in V . Then elements of the set $\{(v_1, w) : w \in W\}$ are not conjugate to any element of set $\{(v_2, w) : w \in W\}$. This follows from Prop. 4.5 using the fact that linear characters of G are the lifts of irreducible characters of $\frac{G}{Z(G)}$. We therefore conclude that the sets $\{(v, w) : w \in W\}$ indexed by non-zero $v \in V$ are mutually disjoint. This way, we divide the non-central elements of G into $|\frac{G}{Z(G)}| - 1$ number of sets each containing $|Z(G)|$ elements.

Theorem 4.7. Let G be a real special 2-group and $q : V := \frac{G}{Z(G)} \rightarrow Z(G) =: W$ be the quadratic map associated to G . Let $v \in V$. If $v \notin \text{rad}(b_{s_*(q)})$ for all non-zero linear maps $s : W \rightarrow \mathbb{F}_2$ then $\{(v, w) : w \in W\}$ is a conjugacy class of G .

Proof. Let $v \notin \text{rad}(b_{s_*(q)})$ for all non-zero linear maps $s : W \rightarrow \mathbb{F}_2$. In view of Prop. 4.5 and Remark 4.6 it is enough to show that $\chi(v, w) = \chi(v, w')$ for all $w, w' \in Z(G)$ and for all irreducible characters χ .

First suppose that χ is linear. Since linear representations of G are precisely the lifts of irreducible representations of $\frac{G}{Z(G)}$, the image of all the elements in the set $\{(v, w) : w \in W\}$ are same under all the linear representations of G .

Now suppose that χ is non-linear. Since $v \notin \text{rad}(b_{s_*(q)})$, by Theorem 4.4(1) $\chi(v, w) = 0$ for all the elements of set $\{(v, w) : w \in W\}$. The result now follows from Prop. 4.5. \square

Theorem 4.8. Let G be a real special 2-group and $q : V := \frac{G}{Z(G)} \rightarrow Z(G) =: W$ be the quadratic map associated to G . Let $v \in V$ and $\mathcal{S}_v := \{s \in \text{Hom}_{\mathbb{F}_2}(Z(G), \mathbb{F}_2) : v \in \text{rad}(b_{s_*(q)})\}$. Then the conjugacy class of element $(v, w) \in G$ is $\{(v, w + w') : s(w') = 0 \text{ for all } s \in \mathcal{S}_v\}$.

Proof. By Remark 4.6, we know that the conjugacy class of (v, w) is a subset of $\{(v, w') : w' \in Z(G)\}$. Let $w_1 \in Z(G)$ be such that $s(w_1) = 1$ for some non-zero $s \in \mathcal{S}_v$. Then

$$\chi_{s,i}(v, w) = \chi_s(\epsilon_s(v), s(w) - \lambda(v) - h_i(v)) = (0, s(w) - \lambda(v) - h_i(v))$$

$$\chi_{s,i}(v, w + w_1) = \chi_s(\epsilon_s(v), s(w + w_1) - \lambda(v) - h_i(v)) = (0, s(w) + 1 - \lambda(v) - h_i(v))$$

It is clear that one of $\chi_{s,i}(v, w)$ and $\chi_{s,i}(v, w + w_1)$ is $\chi_s(0, 0) = 2^l$, while the other one is $\chi_s(0, 1) = -2^l$, where 2^l is the degree of the character $\chi_{s,i}$. Thus $\chi_{s,i}(v, w_1) \neq \chi_{s,i}(v, w_1 + w)$ and it follows from Prop. 4.5 that (v, w) and $(v, w + w_1)$ are not conjugates.

Let $v \in \text{rad}(b_{s_*(q)})$ and $w' \in W$ be such that $s(w') = 0$ for all non-zero linear maps in \mathcal{S}_v . Since linear characters of group G are lifts of irreducible characters of $\frac{G}{Z(G)}$, we

have $\chi((v, w)) = \chi((v, w + w'))$ for all linear characters χ .

Let $\chi_{s,i}$ be a non-linear character and $v \notin \text{rad}(b_{s_*(q)})$, then $\chi_{s,i}((v, w)) = \chi_{s,i}((v, w + w')) = 0$.

Finally we consider the non-linear characters $\chi_{s,i}$ such that $v \in \text{rad}(b_{s_*(q)})$. Then as earlier

$$\chi_{s,i}(v, w) = \chi_s(0, s(w) - \lambda(v) - h_i(v))$$

$$\chi_{s,i}(v, w + w') = \chi_s(0, s(w) + s(w') - \lambda(v) - h_i(v)) = \chi_s(0, s(w) - \lambda(v) - h_i(v))$$

Thus again in this case $\chi_{s,i}((v, w)) = \chi_{s,i}((v, w + w'))$. By Prop. 4.5, (v, w) is conjugate to $(v, w + w')$. \square

We summarize the types of conjugacy classes of special 2-group G in the following table. The notations in the table are same as that of Theorem 4.8.

Table B

Type of element	Conjugacy class
$v \notin \text{rad}(b_{s_*(q)})$ for all $0 \neq s \in \text{Hom}(Z(G), \mathbb{F}_2)$	$\{(v, w) : w \in Z(G)\}$
$v \in \text{rad}(b_{s_*(q)}); s \in \mathcal{S}_v$	$\{(v, w + w') : s(w') = 0 \ \forall \ s \in \mathcal{S}_v\}$

5. EXAMPLE

In this section we demonstrate through an example that the results proved in earlier sections can be used to construct the character table of a real special 2-groups. The example that we consider is that of the group G defined by

$$G = \langle a, b, c, d, f : a^2 = b^2 = (ab)^2 = d, c^2 = (ac)^2 = f, d^2 = f^2 = (bc)^2 = (df)^2 = 1 \rangle.$$

We make following observations about G .

- The center of G is $Z(G) := \langle d, f : d^2 = f^2 = (df)^2 = 1 \rangle$, and the quotient by the center is $\frac{G}{Z(G)} := \langle \bar{a}, \bar{b}, \bar{c} : \bar{a}^2 = \bar{b}^2 = \bar{c}^2 = (\bar{a}\bar{b})^2 = (\bar{a}\bar{c})^2 = (\bar{b}\bar{c})^2 = \bar{1} \rangle$. Both $Z(G)$ and $\frac{G}{Z(G)}$ are elementary abelian 2-groups.
- The group G is a special 2-group as $|G| = 32$ and $Z(G) = \Phi(G) = G' = \langle d, f : d^2 = f^2 = (df)^2 = 1 \rangle$.

We identify $\frac{G}{Z(G)}$ with a 3-dimensional vector space V and $Z(G)$ with a 2-dimensional vector space W over the field \mathbb{F}_2 . Therefore, as a set, the group G gets identified with $V \times W$. Let $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ be a basis of V and $\{f_1 = (1, 0), f_2 = (0, 1)\}$ be a basis of W over \mathbb{F}_2 . The quadratic map $q : V \rightarrow W$ associated to the special 2-group G is given by

$$q(x, y, z) = (x^2 + xy + y^2, z^2 + xz); \quad (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) \in V.$$

We claim that the group G is real. We use Theorem 2.7 to justify this claim. Let $v \in V$. We find $0 \neq v' \in V$ such that $q(v') = q(v + v')$ to show that G is indeed real. The following table explicitly exhibits such $v' \in V$ for a given $v \in V$.

v	v'	$q(v') = q(v + v')$
$(0, 0, 0), (0, 1, 0)$ $(0, 0, 1), (0, 1, 1)$	$(1, 0, 0)$	$(1, 0)$
$(1, 0, 0), (1, 1, 0)$ $(1, 0, 1), (1, 1, 1)$	$(0, 1, 0)$	$(1, 0)$

It follows therefore that G is real.

In the following table we compute the radical $\text{rad}(b_{s_{i*}(q)})$ for each non-zero linear map $s_i : W \rightarrow \mathbb{F}_2$.

Linear map (s)	$s \circ q$	$b_{s*}(q)$	$\text{rad}(b_{s*}(q))$	$ \text{rad}(b_{s*}(q)) $
$s_1(w_1, w_2) = w_1$	$q(x, y, z) = x^2 + xy + y^2$	$b_{s_1*}(q)((x, y, z), (x', y', z'))$ $= xy' + x'y$	$\langle e_3 \rangle$	2
$s_2(w_1, w_2) = w_2$	$q(x, y, z) = z^2 + xz$	$b_{s_2*}(q)((x, y, z), (x', y', z'))$ $= xz' + x'z$	$\langle e_2 \rangle$	2
$s_3(w_1, w_2) = w_1 + w_2$	$q(x, y, z) = x^2 + x(y + z) + (y + z)^2$	$b_{s_3*}(q)((x, y, z), (x', y', z'))$ $= x(y' + z') + x'(y + z)$	$\langle e_2 + e_3 \rangle$	2

We compute the conjugacy classes of G using the results of §4.2.

$\mathcal{C}_1 = \{(0, 0)\}$	$\mathcal{C}_2 = \{(0, f_1)\}$
$\mathcal{C}_3 = \{(0, f_2)\}$	$\mathcal{C}_4 = \{(0, f_1 + f_2)\}$
$\mathcal{C}_5 = \{(e_1, 0), (e_1, f_1), (e_1, f_2), (e_1, f_1 + f_2)\}$	$\mathcal{C}_6 = \{(e_2, 0), (e_2, f_1)\}$
$\mathcal{C}_7 = \{(e_2, f_2), (e_2, f_1 + f_2)\}$	$\mathcal{C}_8 = \{(e_3, 0), (e_3, f_2)\}$
$\mathcal{C}_9 = \{(e_3, f_1), (e_3, f_1 + f_2)\}$	$\mathcal{C}_{10} = \{(e_1, 0)(e_2, 0), (e_1, 0)(e_2, 0)(0, f_1),$ $(e_1, 0)(e_2, 0)(0, f_2), (e_1, 0)(e_2, 0)(0, f_1 + f_2)\}$
$\mathcal{C}_{11} = \{(e_1, 0)(e_3, 0), (e_1, 0)(e_3, 0)(0, f_1),$ $(e_1, 0)(e_3, 0)(0, f_2), (e_1, 0)(e_3, 0)(0, f_1 + f_2)\}$	$\mathcal{C}_{12} = \{(e_2, 0)(e_3, 0), (e_2, 0)(e_3, 0)(0, f_1 + f_2)\}$
$\mathcal{C}_{13} = \{(e_2, 0)(e_3, 0)(0, f_1), (e_2, 0)(e_3, 0)(0, f_2)\}$	$\mathcal{C}_{14} = \{(e_1, 0)(e_2, 0)(e_3, 0), (e_1, 0)(e_2, 0)(e_3, 0)(0, f_1),$ $(e_1, 0)(e_2, 0)(e_3, 0)(0, f_2), (e_1, 0)(e_2, 0)(e_3, 0)(0, f_1 + f_2)\}$

Now, for each non-zero linear map $s : W \rightarrow \mathbb{F}_2$ we compute the regular quadratic forms q_s up to isometry and determine the extraspecial 2-groups G_s associated to these quadratic forms using Remark 2.4.

Linear map (s)	$s \circ q$	q_s	G_s	$ G_s $	Characters	Degree
$s(w_1, w_2) = w_1$	$q(x, y, z) = x^2 + xy + y^2$	$[1, 1]$	$Q_2 \circ D_4$	8	$\chi_{s_1,1}, \chi_{s_1,2}$	2
$s(w_1, w_2) = w_2$	$q(x, y, z) = z^2 + xz$	$[0, 0]$	$D_4 \circ D_4$	8	$\chi_{s_2,1}, \chi_{s_2,2}$	2
$s(w_1, w_2) = w_1 + w_2$	$q(x, y, z) = x^2 + x(y + z) + (y + z)^2$	$[1, 1]$	$Q_2 \circ D_4$	8	$\chi_{s_3,1}, \chi_{s_3,2}$	2

For each non-zero linear map $s_i : W \rightarrow \mathbb{F}_2$ we compute $|\text{rad}(b_{s_i^*(q)})|$ number of non-linear irreducible characters $\chi_{s,j}$ using Prop.4.4. The linear characters of group G are determined using the Remark 3.3.

This summarizes to the following character table of G .

	\mathcal{C}_1	\mathcal{C}_2	\mathcal{C}_3	\mathcal{C}_4	\mathcal{C}_5	\mathcal{C}_6	\mathcal{C}_7	\mathcal{C}_8	\mathcal{C}_9	\mathcal{C}_{10}	\mathcal{C}_{11}	\mathcal{C}_{12}	\mathcal{C}_{13}	\mathcal{C}_{14}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	-1	-1	1	-1	-1	-1	-1
χ_3	1	1	1	1	1	-1	-1	1	1	-1	1	-1	-1	-1
χ_4	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1
χ_5	1	1	1	1	-1	1	1	1	1	-1	-1	1	1	-1
χ_6	1	1	1	1	-1	1	1	-1	-1	-1	1	-1	-1	1
χ_7	1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1
χ_8	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	-1
$\chi_{s_1,1}$	2	-2	2	-2	0	0	0	2	-2	0	0	0	0	0
$\chi_{s_1,2}$	2	-2	2	-2	0	0	0	-2	2	0	0	0	0	0
$\chi_{s_2,1}$	2	2	-2	-2	0	2	-2	0	0	0	0	0	0	0
$\chi_{s_2,2}$	2	2	-2	-2	0	-2	2	0	0	0	0	0	0	0
$\chi_{s_3,1}$	2	-2	-2	2	0	0	0	0	0	0	0	2	-2	0
$\chi_{s_3,2}$	2	-2	-2	2	0	0	0	0	0	0	0	-2	2	0

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INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, KNOWLEDGE CITY, SECTOR-81, MOHALI
140306 INDIA.

E-mail address: `dilpreetkaur@iisermohali.ac.in`

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH, KNOWLEDGE CITY, SECTOR-81, MOHALI
140306 INDIA.

E-mail address: `amitk@iisermohali.ac.in`